people love to count
but it can get technical
fear not, no exams

SYLLABUS (Jan 7 6pm revision) Math 432 Section 39653R
Spring 2021 (online, thanks to COVID)
10am lecture MWF

Professor Richard Arratia,
email: rarratia@usc.edu Please put 432 in the subject line of any email that you send me!

Office hours to be announced.
Grader (reader): to be assigned
Office, which I haven't seen since March 9, 2020: KAP 406C

Text: There is 1 textbook. Invitation to Discrete Mathematics by Matousek and Nesetril. ISBN 0198570422. You should be able to get the paperback for about $70.

GRADING POLICY:
40 percent Homework
50 percent Projects
10 percent Class participation
unspecified small percent Typo-spotting bonus points

Commentary. Homework refers to problems from the text; projects are my own creations. For both, you should present your own work, in your own words. If you do look something up, please name the source! Class participation includes attendance at lectures (— students with a severe time zone difference will be excused from the lecture attendance requirement, but must contact me in advance by email, and make alternate arrangements, such as office hour visits.) Class participation also includes our conversations in lecture and office hours, and one, two, or three individual interviews. For the interviews, I both want to chat informally, to get to know you, and also to hold you to knowing the content of the HW and projects you submitted. I will ask you to discuss/explain some of what you turned in, and perhaps to calculate a variant. If a student fails to seem familiar with the content of what he/she turned in, there will be more extensive follow-up.

Although a grader has not yet been assigned, I imagine that I will soon have a grader, who can then tell me his/her preferences for the mechanics of upload — maybe blackboard, maybe gradescope. I think the HW grading scheme should be two-part, with one score where the reader reports how much of an assignment was done, e.g., 8 of the 10 problems have some response that passes muster at a quick glance, and one score that will use the bulk of the grader's time: correctness, for one or two problems. For each assignment, I will tell the grader which problem(s) to grade carefully, but I won't tell you which, until after the HW is due. A moderate penalty, e.g., 10% per day or per week late, might be offered, to encourage people to not give up if running a bit behind.

Typos (and their cousins, spelling errors, and thinkos) are, unfortunately, all too common when I write or typeset — and when one occurs in lecture, or in one of my handouts – I will show my gratitude, via bonus points, to the first student who points it out! These bonus points are beyond the basic 40% + 50% + 10% scheme. No penalty for calling attention to something that was already correct; when in doubt, if you find something confusing, you should speak up quickly.

Topics will include, perhaps with some omissions due to time pressure: basic counting, inclusion-exclusion, recursions, ordinary generating functions, exponential generating functions, Stirling numbers of the first and second kind, the four-fold way, the twelve-fold way. Unified somewhat by consideration of labelled versus unlabelled structures: more on integer partitions, set partitions, permutations and cycle structure, simple graphs, trees, directed graphs, the Polya-Burnside theory. Some other special numbers, including Fibonacci numbers, Eulerian numbers, Bernoulli numbers. Maybe: (over an alphabet of size $q \geqslant 2$) circular words, necklaces, Lyndon words, irreducible polynomials over the field with q elements, especially $q = 2$, and de Bruijn sequences.

Possibly — depending on feedback from students (and grader) — some of the HW will be numerical, of a size requiring computer assistance. For example, let $a(n, k)$ be the number of lists of length n where the elements are $\{1, 2, \ldots, k\}$ — with each of the k possibilites occurring at least once. For example, $a(4, 2) = 14 = 4 + 6 + 4$. [There are $\binom{4}{2}$=6 ways, with 2 places for 1s, and the remaining places for the 2s, and there are 4 ways to choose one place for a 1 with the other 3 places for the 2, and 4 ways to pick 3 places for 1s with the other one place for the 2.] We will teach two ways to give a formula for $a(n, k)$ in general. But to give the value of $a(20, 10)$, by either formula, might require a computer. And I might require: write a barehands program, don't simply ask `https://www.wolframalpha.com` .

The syllabus proper is the material above; the material below may be skipped, or viewed as a bonus lecture.

TMI: The ISBN of our textbook is 0198570422; this is not the ISBN number, because the final N in ISBN is for number. See `N Is a Number: A Portrait of Paul Erdos` — `https://www.imdb.com/title/tt0125425/`. In the preceding sentence, I had to typeset the ő in the name Erdős using a plain letter 'oh'; it is devilishly hard to get printed the 27th letter of the 44 letter Hungarian alphabet
`https://en.wikipedia.org/wiki/Hungarian_alphabet` , in the context of typesetting this syllabus, even in Latex, — https://en.wikipedia.org/wiki/LaTeX — the favorite typesetting system for mathematicians: see https://tex.stackexchange.com/questions/276534 or

https://tex.stackexchange.com/questions/44335

Fun TMI: The top of this syllabus has a preferred melody, written by Karl Suessdorf `https://en.wikipedia.org/wiki/Karl_Suessdorf` . The best way to learn the melody is to listen to a good performance such as https://www.youtube.com/watch?v=RuoU7XW2ops or `https://www.youtube.com/watch?v=cyyRi3l7E_k` ; please listen, and then try to sing the top of my syllabus.

A good word for previous years' texts: In 2020, I had 2 required books; a student might have purchased the pair, new, for \$(20+60)=\$80. (Do not buy these. They are good values, and I will tell you about some other good values later.) First was the basic and highly competent book by Balakrishnan, 'Schaum's Outline of Theory and Problems of Combinatorics including concepts of Graph Theory'; only one edition exists, and it is paperback. The other 2020 text was the brilliant, and sometimes maddeningly brilliant, pun-filled 'Concrete Mathematics: A Foundation for Computer Science', by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. A vey interesting web page devoted to the book is `https://www-cs-faculty.stanford.edu/~knuth/gkp.html` and this includes an errata, where the number of items is, approximately, pardon my spanglish, 'sin cuenta' — fifty. Ten years ago, in 2011, my 432 text book was Foundations of Combinatorics with Applications. Dover, 2006, Ed Bender and S. Gill Williamson. And, I told my students: Available online at `http://math.ucsd.edu/∼ebender/CombText/` ; I recommend that you spend the 20 dollars to have a printed bound copy, and also download an electronic copy. This text covers about *three* times as much material as could reasonably be covered in one semester for upper division undergraduate math majors, and we will be reasonable. We will emphasize the material from parts I and IV, Counting, and Generating Functions.

Guiding course philosophy Not only is counting fun, it is often the best way to clarify communication between mathematicians.

Communication, even between mathematicians, is really hard. A simple English word may have several mathematical meanings — depending on which community of mathematicans you are talking to. For example what does the word **graph** mean? Even in the context of a combinatorics class, there are at least 10 meanings!

1. For a function $f : A \to B$, the **graph** is (a picture of) the subset of the Cartesian product $A \times B$ consisting of all ordered pairs (a, b) for $a \in A$ and $b = f(a)$; more generally, for a relation R, the graph is (a picture of) the subset of the Cartesian product $A \times B$ consisting of all ordered pairs (a, b) for which aRb. When A and B are finite sets, with $|A| = n$ and $|B| = k$, there are n^k functions f, and 2^{nk} relations R; perhaps without studying the careful discussion in sections 1.4 and 1.5 of our text, you can deduce what Definition 1.4.1 and Definition 1.5.1 say about functions and relations, just from the counts, n^k functions f, and 2^{nk} relations R. I will be very happy if my students get to the place where they are comfortable answering two questions: what basic objects are counted by n^k and 2^{nk}? Alas, for an online semester, where all exams are open book, those questions become too easy.

When A and B are both $\mathbb{R}$, this notion of **graph**, drawn with Cartesian coordinates, is what you study in calculus, and what you represent, approximately, with your **graphing calculator**. But for the case with $|A| = n$ and $|B| = k$, with $A, B \subset \mathbb{R}$, the picture has n isolated points, rather dull, (Figure 1.3 on p. 33 tries to *doll up* the picture by replacing points with squares,) and people usually don't bother with the picture.

(Not combinatorics, though loosely related.) For one of the smallest infinite situations, namely the case where $A = \{0, 1\}$ is a set of size 2 and $B = \mathbb{N} := \{1, 2, 3, \ldots\}$ is the set of natural numbers — whose size is the smallest infinite cardinal, the question of **counting**, how many functions exist, is very subtle — is it the second smallest infinite cardinal, or not? See for example `https://www.storyofmathematics.com/20th_cohen.html` and `https://en.wikipedia.org/wiki/Paul_Cohen` . An even larger cardinal is the number of functions $f : A \to B$ when A and B are both $\mathbb{R}$.

2. The graph of a function $f : A \to B$ is the subset of the Cartesian product $A \times B$ consisting of all ordered pairs (a, b) for $a \in A$ and $b = f(a)$ — **with the ordered pair** (a, b) **drawn as an arrow**, from a to b. Similarly for relations. (Which is the head, and which is the tail? I believe both answers!) Some people say *digraph* instead of *graph*, to emphasize that the direction of an edge matters. When A and B are finite sets, with $|A| = n$ and $|B| = k$, there are n^k functions; the case $A = B = [n] := \{1, 2, \ldots, n-1, n\}$ is of great interest. We will study these, and call them *endofunctions* of size n, or *random mappings*, or, to emphasize the directed arrow picture, random mapping digraphs.

3. In general, when a combinatorist thinks of graphs, there are points connected by edges, so that it is natural so also consider paths. I believe that these graphs come in $8 = 2^3$ flavors, according to the choice of

   (a) Does edge ab mean (a, b) or $\{a, b\}$, i.e., are the edges directed, or undirected?

   (b) Loops: can there be a *loop* edge, where both the endpoints are the same vertex?

   (c) Multiple edges: given two vertices a, b, can there be more than one ab edge?

4. **Simple graph** or *simple undirected graph* (our text, p. 97) refers to only one of the 8 cases in item 3) above, specifically, the one where the answers are: undirected, no loops, no multiple edges. For endofunctions, that is random mappings, the answers are: directed, loops allowed, no multiple edges — and the special feature which makes a such a directed graph into a the digraph of a **function**, rather than the digraph of a relation (sections 1.4 and 1.5 of our text) is that every vertex has out-degree exactly 1.

Review question 1: I said "there are at least 10 meanings" for "graph". How many did I list, above? We should argue about the correct answer, 4, 9, 10, or 11, in lecture. Review question 2: will this be on the final exam?

When an English phrase is ambiguous, one should try to clarify: list the possible interpretations, then for each, under restrictions such as 'total size n, with k parts', give a count of how many. For example, if someone says "Consider graphs with n vertices", then you can clarify by asking "How many are there?" If the answer is $2^{\binom{n}{2}}$, then your someone was thinking of simple graphs, as in item 3 above.

Counting is not the be-all and end-all of combinatorial understanding. There is a beautiful self-fulfilling example: enumerate the meanings of *enumerate*.
An official source is https://www.merriam-webster.com/dictionary/enumerate

Enumeration has two dictionary definitions.
1. To find the total number of objects,
2. To list them in order.

I find it very interesting to think about both senses of enumeration.

Silly observation: many mathematicians, especially the computer-science oriented ones, would have written the list as
0. To find the total number of objects,
1. To list them in order.

Serious observation: many mathematicians, scientists, and programmers like to start counting from 0; this causes errors when inter-operating with people who start counting from 1. It is similar to the metric units/english units conflict, see
https://www.britannica.com/topic/Imperial-unit and
https://www.latimes.com/archives/la-xpm-1999-oct-01-mn-17288-story.html

For example, on $n = 20$, it is fairly easy to determine which permutation is one-trillionth, in lexicographic order, but quite hard to determine which derangement (i.e., fixed-point-free permutation) is one-trillionth, in lexicographic order. An automatic A, plus \$20, given to the first student to answer — with proof — since plausible guesses, with chance around one in a million of being correct, are not hard to generate — no credit for finding and paying the USC graduate student who solved this for me last year!